U6H2

Center of Mass of Uniform and non-uniform objects using calculus

$$x_{COM} = \frac{\int x\,dm}{\int dm}; y_{COM} = \frac{\int y\,dm}{\int dm}$$

Linear mass density is defined as $\lambda = \frac{dm}{dx}$ = mass per unit length

Surface mass density is defined as $\sigma = \frac{dm}{dA}$ = mass per unit area

Volume density (good ole density) $\rho = \frac{dm}{dV}$ = mass per unit volume

1. Find the center of mass of a rod of uniform density λ and length L.

$$Com_x = \frac{\int dm\,x}{\int dm} = \frac{\int \lambda x\,dx}{\int \lambda\,dx} = \frac{\cancel{\lambda}\int_0^L x\,dx}{\cancel{\lambda}\int_0^L dx} = \frac{\int_0^L x\,dx}{\int_0^L dx} = \frac{\left[\frac{x^2}{2}\right]_0^L}{\left[x\right]_0^L} = \frac{\left(\frac{L^2}{2} - \frac{0^2}{2}\right)}{(L-0)}$$

$$= \boxed{\frac{L}{2}}$$

2. Find the center of mass of a rod of length L=1m, whose density varies with location as λ=2x+3, being least dense at one end.

$$\frac{\int dm\,x}{\int dm} = \frac{\int \lambda x\,dx}{\int \lambda\,dx} = \frac{\int (2x+3)\,x\,dx}{\int (2x+3)\,dx} = \frac{\int_0^1 2x^2+3x\,dx}{\int 2x+3\,dx} = \frac{\left[\frac{2x^3}{3} + \frac{3x^2}{2}\right]_0^L}{\left[x^2+3x\right]_0^L}$$

$$= \frac{\frac{2L^3}{3} + \frac{3L^2}{2}}{L^2+3L}$$

$$= \frac{4L^3 + 9L^2}{6(L^2+3L)}$$

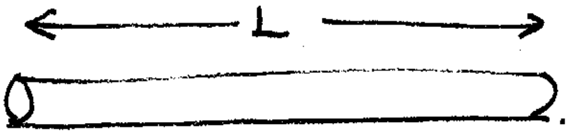
$$= \frac{4+9}{6(4)} = \frac{13}{24}$$

3. Find the COM of a uniform wire in the shape of a semicircle of radius R.

4. Find the COM of a uniform semicircular disk of radius R

$\boxed{\text{U6H2}}$

1.



$$\lambda = \frac{\text{mass}}{\text{length}} . \longrightarrow \text{mass} = \lambda \cdot \text{length}$$

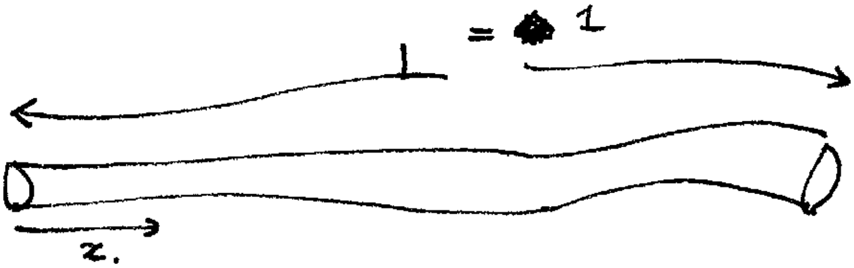
$$\therefore \int_0^L \lambda \cdot dx \cdot x = \int_0^L \lambda \cdot x \cdot dx .$$

$$x = \frac{\lambda \left[\frac{1}{2}x^2\right]_0^L}{\lambda \left[x\right]_0^L}$$

$$= \frac{\frac{1}{2}L^2}{L} = \frac{1}{2}L$$

UGH2

2.



$\lambda = 2x + 3$

$$\int_0^1 (2x+3)\,dx \cdot x_{CM} = \int_0^1 (2x+3)\cdot x \cdot dx.$$

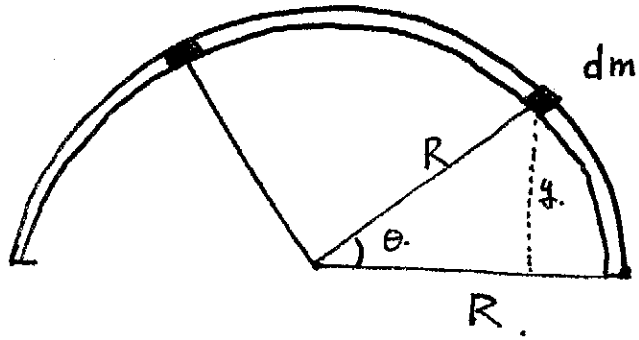
$$x_{CM} = \frac{\left[\frac{2}{3}x^3 + \frac{3}{2}x^2\right]_0^1}{\left[x^2 + 3x\right]_0^1}$$

$$= \frac{\frac{2}{3} + \frac{3}{2}}{1+3} = \frac{\frac{4+9}{6}}{4}$$

$$= \frac{13}{24} \; m$$

U6H2

3.



$dm = \lambda \cdot ds = \lambda \cdot R \cdot d\theta$

M = Total Mass.

x-direction $\rightarrow$ sym.

y-direction.

$$M \cdot y_{CM} = \int_0^\pi R \cdot \sin\theta \cdot dm = \int_0^\pi R \cdot \sin\theta \cdot \lambda \cdot R \cdot d\theta$$

$$= \lambda \cdot R^2 \int_0^\pi \sin\theta \cdot d\theta$$

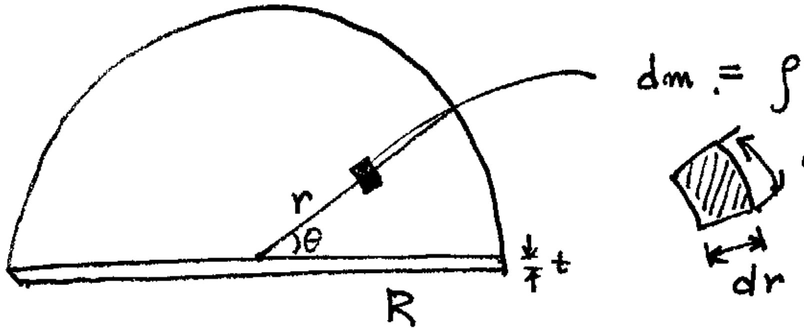
$$* \quad M = \int_0^\pi dm = \lambda R \cdot \int_0^\pi d\theta = \lambda R \pi$$

$$\therefore \cancel{\lambda} R \pi \cdot y_{CM} = \cancel{\lambda} R^{\cancel{2}} \left[-\cos\theta\right]_0^\pi$$

$$y_{CM} = \frac{R\,(1-(-1))}{\pi} = \frac{2R}{\pi}$$

U6H2

4.



$dm = \rho \cdot ds \cdot dr \cdot t$

$= \rho \cdot t \cdot r \cdot d\theta \cdot dr$

x – direction sym.

y – direction.

$$\int_0^R \int_0^\pi \rho \cdot t \cdot r \cdot d\theta \, dr \cdot y_{CM} = \int_0^R \int_0^\pi \rho \cdot t \cdot r \cdot r \cdot \sin\theta \, d\theta \cdot dr$$

$$\cancel{\rho \cdot t} \int_0^R r \left\{ \int_0^\pi d\theta \right\} dr \cdot y_{CM} = \cancel{\rho \cdot t} \int_0^R r^2 \int_0^\pi \sin\theta \, d\theta \, dr$$

$$y_{CM} = \frac{\int_0^R r^2 \left[-\cos\theta\right]_0^\pi dr}{\int_0^R r \, \pi \, dr} = \frac{2\left[\frac{1}{3} r^3\right]_0^R}{\pi \left[\frac{1}{2} R^2\right]_0^R}$$

$$= \frac{2 \, \frac{1}{3} R^{\cancel{3}}}{\pi \, \frac{1}{2} \cancel{R^2}} = \frac{4R}{3\pi}$$